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NONLINEAR STOCHASTIC CREEP PROBLEM FOR AN INHOMOGENEOUS PLANE WITH THE DAMAGE TO THE MATERIAL TAKEN INTO ACCOUNT

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An analytical method for the solution of two-dimensional nonlinear creep problems is developed using as an example the biaxial extension of a plane from a stochastically inhomogeneous material with damage accumulation and the third stage of creep taken into account. The governing creep relation is adopted in accordance with the energetic version of the nonlinear theory of viscous flow. The stochasticity of the material is defined by two random functions of coordinates. Formulas for calculating the stress variance are obtained.

Key words: *stochastic inhomogeneity, steady-state creep, damage, random function, stress variance.*

At present, adequately thorough studies of structurally inhomogeneous media using the theory of random functions have been performed only for linearly elastic media [1]. Under creep conditions, the development of analytical methods for solving stochastic boundary-value problems is significantly complicated, primarily because of the physical and stochastic nonlinearity of the governing equations. One-dimensional stochastic problems of steady-state creep (for example, a tube subjected to internal pressure) can be solved with any degree of accuracy using the small parameter method [2]. As regards plane and spatial creep problems, they have been solved only as a first approximation using steady-state creep theory [3–5]. Analytical methods for solving stochastic boundary-value problems with damage accumulation and the third stage of creep taken into account have not been developed.

Let the components of the nominal-stress tensor σ_{ij} satisfy the equilibrium equations

$$\sigma_{ij,j} = 0 \qquad (i, j = 1, 2), \tag{1}$$

and let the components of the strain-rate tensor $\dot{p}_{ij}$ satisfy the condition

$$\Lambda_{ij}\Lambda_{kl}\dot{p}_{jk,il} = 0, \tag{2}$$

which is obtained by differentiation of the strain compatibility equation with respect to time. Here Λ_{ij} is a unit antisymmetric pseudotensor. The summation from 1 and 2 is performed over repeated indices.

Equations (1) and (2) are closed by the stochastic governing relations of the nonlinear theory of viscous flow (steady-state creep) [3]:

$$\dot{p}_{ij} = c\bar{s}^{n-1}(\bar{\sigma}_{ij} - (1/3)\delta_{ij}\bar{\sigma}_{mm})(1 + \alpha_1 U_1(x_1, x_2)) \qquad (i, j = 1, 2). \tag{3}$$

Here $\bar{s}$ is the stress intensity:

$$\bar{s}^2 = (3\bar{\sigma}_{ij}\bar{\sigma}_{ij} - \bar{\sigma}_{ii}\bar{\sigma}_{jj})/2,$$

$\bar{\sigma}_{ij}$ are the components of the true-stress tensor, δ_{ij} is the Kronecker delta, $U_1(x_1, x_2)$ is a random homogeneous function that describes the rheological characteristics of the material with expectation $\langle U_1 \rangle = 0$ and variance $\langle U_1^2 \rangle = 1$, and c, n, and α_1 are material constants.

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The governing equations with the third stage of creep taken into account are written using the energetic version of steady-state creep theory [6], according to which the true stresses $\bar{\sigma}_{ij}$ are related to the nominal stresses σ_{ij} by the formula

$$\bar{\sigma}_{ij} = \sigma_{ij}(1+\omega), \tag{4}$$

where $\omega(t, x_1, x_2)$ is a scalar damage parameter that describes damage accumulation with time (at each point of the material) due to creep and satisfies the kinetic equation

$$\dot{\omega} = b(1+\alpha_2 U_2(x_1,x_2))\bar{\sigma}_{ij}\dot{p}_{ij}. \tag{5}$$

Here b and α_2 are material constants, $U_2(x_1, x_2)$ is a random homogeneous function that describes the stochastic damage to the material with expectation $\langle U_2\rangle = 0$ and variance $\langle U_2^2\rangle = 1$.

The problem formulated here is physically and statistically nonlinear, and, hence, it is solved approximately for the nominal stresses σ_{ij} by linearization using the small parameter method.

Using (4), we write relations (3) as

$$\dot{p}_{ij} = \dot{r}_{ij}(1+\omega)^n, \tag{6}$$

where

$$\dot{r}_{ij} = cs^{n-1}(\sigma_{ij} - (1/3)\delta_{ij}\sigma_{mm})(1+\alpha_1 U_1(x_1,x_2)). \tag{7}$$

Here s is the nominal-stress intensity. Formula (7) specifies the creep law for $\omega(t) \equiv 0$. In view of (6), Eq. (5) for the damage parameter ω is written as

$$\dot{\omega} = b(1+\alpha_2 U_2)\dot{r}_{ij}(1+\omega)^n. \tag{8}$$

Integrating Eq. (8) subject to the initial condition $\omega(0) = 0$, for the quantity $(1+\omega)^n$, we have

$$(1+\omega)^n = 1 \Big/ \Big(1 - bn(1+\alpha_2 U_2)\int\limits_0^t \sigma_{mn}\dot{r}_{mn}\,d\tau\Big).$$

Linearization of the right side of the last relation yields

$$(1+\omega)^n \approx 1 + bn(1+\alpha_2 U_2)\int\limits_0^t \sigma_{mn}\dot{r}_{mn}\,d\tau. \tag{9}$$

In view of (9), relation (6) becomes

$$\dot{p}_{ij} = \dot{r}_{ij}\Big(1 + bn(1+\alpha_2 U_2)\int\limits_0^t \sigma_{mn}\dot{r}_{mn}\,d\tau\Big). \tag{10}$$

Let the nominal-stress tensor be represented as the sum of the deterministic term σ_{ij}^0 and the fluctuation σ_{ij}^*:

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^*, \qquad \langle\sigma_{ij}\rangle = \sigma_{ij}^0.$$

Linearization of relation (7) was performed in [3] using the condition that σ_{11}^0 and σ_{22}^0 are constant and $\sigma_{12}^0 = 0$:

$$\begin{gathered}\dot{r}_{11} = \dot{r}_{11}^0 + (1/3)cs_0^{n-1}(2\sigma_{11}^* - \sigma_{22}^* + (\sigma_{11}^* l_1 + \sigma_{22}^* l_2)k_1 + \alpha_1 U_1 l_1),\\ \dot{r}_{22} = \dot{r}_{22}^0 + (1/3)cs_0^{n-1}(2\sigma_{22}^* - \sigma_{11}^* + (\sigma_{11}^* l_1 + \sigma_{22}^* l_2)k_2 + \alpha_1 U_1 l_2),\\ \dot{r}_{12} = cs_0^{n-1}\sigma_{12}^*.\end{gathered} \tag{11}$$

Here

$$\dot{r}_{11}^0 = (1/3)cs_0^{n-1}l_1, \qquad \dot{r}_{22}^0 = (1/3)cs_0^{n-1}l_2, \qquad s_0^2 = (\sigma_{11}^0)^2 + (\sigma_{22}^0)^2 - \sigma_{11}^0\sigma_{22}^0,$$

$$l_1 = 2\sigma_{11}^0 - \sigma_{22}^0, \qquad l_2 = 2\sigma_{22}^0 - \sigma_{11}^0, \qquad k_i = (n-1)l_i/(2s_0^2).$$

Each of the quantities $\dot{p}_{ij}$, σ_{ij}, and $\dot{r}_{ij}$ in relation (10) is represented as a deterministic part and a random part:

$$\dot{p}^0_{ij} + \dot{p}^*_{ij} = (\dot{r}^0_{ij} + \dot{r}^*_{ij})\Big(1 + bn(1+\alpha_2 U_2)\int\limits_0^t (\sigma^0_{mn} + \sigma^*_{mn})(\dot{r}^0_{mn} + \dot{r}^*_{mn})\, d\tau\Big). \tag{12}$$

Removing the parentheses on the right of expression (12) with the use of relation (11), collecting terms, and discarding terms of the second and third orders of smallness, for the strain rate fluctuations we obtain

$$\dot{p}^*_{11} = A^2 bnl_1\Big(B\int\limits_0^t \sigma^*_{11}\, d\tau + C\int\limits_0^t \sigma^*_{22}\, d\tau\Big) + A^2 bnl_1\alpha_1 U_1(\sigma^0_{11}l_1 + \sigma^0_{22}l_2)t$$

$$+ A^2 bnl_1\alpha_2 U_2(\sigma^0_{11}l_1 + \sigma^0_{22}l_2)t + A(\sigma^*_{11}(2 + l_1k_1) + \sigma^*_{22}(-1 + l_1k_2) + \alpha_1 U_1 l_1)(1 + A_1 t), \tag{13}$$

$$\dot{p}^*_{22} = A^2 bnl_2\Big(B\int\limits_0^t \sigma^*_{11}\, d\tau + C\int\limits_0^t \sigma^*_{22}\, d\tau\Big) + A^2 bnl_2\alpha_1 U_1(\sigma^0_{11}l_1 + \sigma^0_{22}l_2)t$$

$$+ A^2 bnl_2\alpha_2 U_2(\sigma^0_{11}l_1 + \sigma^0_{22}l_2)t + A(\sigma^*_{11}(-1 + l_1k_2) + \sigma^*_{22}(2 + l_2k_2) + \alpha_1 U_1 l_2)(1 + A_1 t),$$

$$\dot{p}^*_{12} = 3A\sigma^*_{12}(1 + A_1 t),$$

where

$$A = (1/3)cs_0^{n-1}, \qquad A_1 = Abn(\sigma^0_{11}l_1 + \sigma^0_{22}l_2),$$

$$B = l_1 + \sigma^0_{11}(2 + l_1k_1) + \sigma^0_{22}(-1 + l_1k_2), \qquad C = l_2 + \sigma^0_{11}(-1 + l_2k_1) + \sigma^0_{22}(2 + l_2k_2).$$

Substitution of (13) into the compatibility equation for the strain-rate fluctuations

$$\dot{p}^*_{11,22} + \dot{p}^*_{22,11} - 2\dot{p}^*_{12,12} = 0$$

yields the relation

$$Abn\Big(l_1 B\int\limits_0^t (\sigma^*_{11,11} + \sigma^*_{11,22})\, d\tau + l_2 C\int\limits_0^t (\sigma^*_{22,11} + \sigma^*_{22,22})\, d\tau\Big)$$

$$+ Abnt\alpha_1(l_1U_{1,22} + l_2U_{1,11})(\sigma^0_{11}l_1 + \sigma^0_{22}l_2) + (\sigma^*_{11,22}(2 + l_1k_1) + \sigma^*_{22,22}(-1 + l_1k_2)$$

$$+ \sigma^*_{11,11}(-1 + l_2k_1) + \sigma^*_{22,11}(2 + l_2k_2) + \alpha_1(U_{1,22}l_1 + U_{1,11}l_2))(1 + A_1 t)$$

$$+ Abnt\alpha_2(l_1U_{2,22} + l_2U_{2,11})(\sigma^0_{11}l_1 + \sigma^0_{22}l_2) - 6\sigma^*_{12,12}(1 + A_1 t) = 0. \tag{14}$$

This equation should be supplemented by the equilibrium equations for the stress fluctuations

$$\sigma^*_{ij,j} = 0. \tag{15}$$

If we introduce the stress function F for the nominal-stress tensor fluctuations

$$\sigma^*_{11} = F_{,22}, \qquad \sigma^*_{22} = F_{,11}, \qquad \sigma^*_{12} = -F_{,12}, \tag{16}$$

then, instead of system (14), (15), we obtain the following differential equation for the function F:

$$Abn\Big(l_1 B\int\limits_0^t (F_{,2222} + F_{,1122})\, d\tau + l_2 C\int\limits_0^t (F_{,1111} + F_{,1122})\, d\tau\Big)$$

$$+ Abnt\alpha_1(l_1U_{1,22} + l_2U_{1,11})(\sigma^0_{11}l_1 + \sigma^0_{22}l_2) + (F_{,1111}(2 + l_2k_2) + 2F_{,1122}(2 + l_1k_2)$$

$$+ F_{,2222}(2 + l_1k_1) + \alpha_1 U_{1,22}l_1 + \alpha_1 U_{1,11}l_2)(1 + A_1 t)$$

$$+ Abnt\alpha_2(l_1U_{2,22} + l_2U_{2,11})(\sigma^0_{11}l_1 + \sigma^0_{22}l_2) = 0. \tag{17}$$

Because of the computational difficulties encountered in the solution of Eqs. (17), we shall further analyze only the case of uniform extension ($\sigma^0_{11} = \sigma^0_{22} = \sigma^0$). Then, Eq. (17) becomes

$$\frac{n+3}{2}\,(F_{,1111} + 2F_{,1122} + F_{,2222})(1 + A_1 t) + \frac{A_1}{2}\,(n+1)\int\limits_0^t (F_{,1111} + 2F_{,1122} + F_{,2222})\,d\tau$$

$$= -(1 + 2A_1 t)\alpha_1\sigma^0(U_{1,11} + U_{1,22}) - A_1 t\alpha_2\sigma^0(U_{2,11} + U_{2,22}). \tag{18}$$

Let the functions $U_k(x_1, x_2)$ ($k = 1, 2$), which are used to specify the random perturbation fields for the mechanical properties of the material, be homogeneous and isotropic. Then, they can be represented as the Fourier–Stieltjes stochastic integrals [6]

$$U_k(x_1, x_2) = \iint\limits_{-\infty}^{+\infty} e^{i(q_1x_1+q_2x_2)}\; d\varphi_k(q_1, q_2) \qquad (k = 1, 2), \tag{19}$$

for the random differentials, the following relation holds:

$$\langle d\varphi_k(q_1, q_2)\overline{d\varphi_k(q'_1, q'_2)}\rangle = S_k(q_1, q_2)\delta(q_1 - q'_1)\delta(q_2 - q'_2)\, dq_1\, dq_2\, dq'_1\, dq'_2.$$

Here $S_k(q_1, q_2)$ is the spectral density of the field U_k and $\delta(x)$ is the Dirac delta function; the bar denotes complex conjugation (the summation is not performed over the index k).

Since the random fields of microinhomogeneities $U_k(x_1, x_2)$ are rapidly oscillating, the solution of the linearized problem (18) will be homogeneous and it can be sought in the form

$$F = \iint\limits_{-\infty}^{+\infty} e^{i(q_1x_1+q_2x_2)}(b_1(q_1, q_2, t)\, d\varphi_1(q_1, q_2) + b_2(q_1, q_2, t)\, d\varphi_2(q_1, q_2)), \tag{20}$$

where $b_k(q_1, q_2, t)$ ($k = 1, 2$) are unknown weight functions.

Substitution of the representation (19), (20) into relation (18) yields the following two equations for the weight functions $b_k(q_1, q_2, t)$:

$$\frac{n+3}{2}\, b_1 q^2(1 + A_1 t) + \frac{A_1}{2}\,(n+1)q^2\int\limits_0^t b_1\, d\tau = (1 + 2A_1 t)\alpha_1\sigma^0; \tag{21}$$

$$\frac{n+3}{2}\, b_2 q^2(1 + A_1 t) + \frac{A_1}{2}\,(n+1)q^2\int\limits_0^t b_2\, d\tau = A_1 t\alpha_2\sigma^0, \qquad q^2 = q_i q_i. \tag{22}$$

Let us consider the solution of Eq. (21). The change of variable $b_1(t) = \dot{x}(t)$ reduces Eq. (21) to the first-order linear differential equation

$$\frac{n+3}{2}\,\dot{x}(t)q^2(1 + A_1 t) + \frac{A_1}{2}\,(n+1)q^2 x(t) = (1 + 2A_1 t)\alpha_1\sigma^0,$$

whose solution has the form

$$x(t) = C(1 + A_1 t)^{-(n+1)/(n+3)} + \frac{2\alpha_1\sigma^0(1 + 2A_1 t)}{A_1(n+1)q^2} - \frac{2\alpha_1\sigma^0(1 + A_1 t)(n+3)}{A_1(n+1)(n+2)q^2} \tag{23}$$

(C is an arbitrary constant). Differentiation of solution (23) yields

$$b_1(t) = -CA_1\,\frac{n+1}{n+3}\,(1 + A_1 t)^{-2(n+2)/(n+3)} + \frac{2\alpha_1\sigma^0}{(n+2)q^2}.$$

The constant C can be found using the initial condition

$$b_1(0) = \frac{2\alpha_1\sigma^0}{(n+3)q^2},$$

which is calculated in accordance with the results of [3] for $\omega = 0$. As a result, the weight function $b_1(t)$ is defined by the formula

TABLE 1

α_1	$\sqrt{D_{11}(0)}/\sigma^0 \cdot 10^2$
0.1	1.98
0.2	3.95
0.3	5.93
0.4	7.90
0.5	9.88

TABLE 2

α_1	$\sqrt{D_{11}(1000)}/\sigma^0 \cdot 10^2$				
	$\alpha_2 = 0.1$	$\alpha_2 = 0.2$	$\alpha_2 = 0.3$	$\alpha_2 = 0.4$	$\alpha_2 = 0.5$
0.1	3.30	5.20	7.30	9.50	11.80
0.2	5.25	6.60	8.40	10.30	12.50
0.3	7.43	8.40	9.90	11.70	13.40
0.4	9.70	10.40	11.70	13.20	14.90
0.5	12.00	12.60	13.70	15.00	16.50

$$b_1(t) = -\frac{2\alpha_1\sigma^0(1 + A_1 t)^{-2(n+2)/(n+3)}}{(n+2)(n+3)q^2} + \frac{2\alpha_1\sigma^0}{(n+2)q^2}. \tag{24}$$

Equation (22) is solved similarly. Its solution has the form

$$b_2(t) = \frac{2\alpha_2\sigma^0}{(n+2)q^2}\Big(1 - (1 + A_1 t)^{-2(n+2)/(n+3)}\Big). \tag{25}$$

Thus, according to Eqs. (9), (24), and (25), the components of the tensor of nominal stress fluctuations are calculated by the formula

$$\sigma^*_{kl} = \iint\limits_{-\infty}^{+\infty} \mathrm{e}^{i(q_1x_1+q_2x_2)}(c_{kl}(q_1, q_2, t)\, d\varphi_1(q_1, q_2) + d_{kl}(q_1, q_2, t)\, d\varphi_2(q_1, q_2)), \tag{26}$$

where $c_{11} = -q_2^2 b_1$, $c_{12} = q_1q_2b_1$, $c_{22} = -q_1^2 b_1$, $d_{11} = -q_2^2 b_2$, $d_{12} = q_1q_2b_2$, and $d_{22} = -q_1^2 b_2$.

Solution (26) can be used to study the random stress field in the case of creep of a flat plate. For rapidly varying random fields of microinhomogeneities U_k, the boundaries have a fairly small effect on the stress-strain state in the internal region. Therefore, far from the boundary of the plate, the stress state is defined by formula (26). Near the boundary on which the deterministic boundary conditions are specified, it is necessary to perform additional studies.

To determine the variances of the random stress field, we assume that the creep and damage accumulation processes have independent effects on the probabilistic characteristics of the stresses. Under this condition, the stress variances are defined by the formula

$$D_{kl}(t) = D(\sigma_{kl}) = \iint\limits_{-\infty}^{+\infty} (S_1(q_1,\, q_2)c_{kl}^2(q_1, q_2, t) + S_2(q_1,\, q_2)d_{kl}^2(q_1, q_2, t))\, dq_1\, dq_2. \tag{27}$$

The spectral density S_i of the isotropic scalar field U_i depends only on the wave-vector modulus $q_0 = \sqrt{q_1^2 + q_2^2}$, and for the variance, the following equality holds [7]:

$$D_{U_i} = 2\pi \int\limits_0^\infty S_i(q_0) q_0\, dq_0 = 1. \tag{28}$$

Passing to the polar coordinates $q_1 = q_0 \cos\varphi$ and $q_2 = q_0 \sin\varphi$ in the integral (27) and integrating it subject to (28), we obtain the following equalities for the stress variance:

$$D_{11}(t) = D_{22}(t) = 3D_{12}(t)$$

$$= \frac{3\alpha_1^2(\sigma^0)^2}{2(n+2)^2}\Big(1 - \frac{(1 + A_1 t)^{-2(n+2)/(n+3)}}{n+3}\Big)^2 + \frac{3\alpha_2^2(\sigma^0)^2}{2(n+2)^2}\Big(1 - (1 + A_1 t)^{-2(n+2)/(n+3)}\Big)^2.$$

As an example, we consider uniform extension of a plate of 12Kh18N10T steel at a temperature $T = 1123$ K and a stress $\sigma^0 = 39.24$ MPa and for the following parameters of the governing relations: $n = 3.2$, $c = 6.67 \cdot 10^{-9}$ MPa$^{-3.2} \cdot$ h^{-1}, and $b = 0.141$. These data are taken from [8]. Table 1 gives values of the coefficient of variation $\sqrt{D_{11}(0)}/\sigma^0$ at $t = 0$ and for various values of α_1, which corresponds to the case of steady-state creep ignoring damage accumulation. Table 2 gives values of $\sqrt{D_{11}(1000)}/\sigma^0$ at $t = 1000$ h versus the parameters α_1 and α_2.

From Tables 1 and 2, it follows that the coefficient of variation $\sqrt{D_{11}(t)}/\sigma^0$ can be significant and, depending on the parameters α_1 and α_2 and time t, it can be several times higher than the corresponding value in the case of steady-state creep ignoring damage accumulation. In other words, in the third stage of creep, the stress fluctuations change (increase) with time, which provides a theoretical explanation for the experimentally observed increase in the variation of the creep strain in the softening stage compared to the variation in the stage of steady-state creep.

REFERENCES

1. V. A. Lomakin, *Elasticity theory for Inhomogeneous Solids* [in Russian], Izd. Mosk. Univ., Moscow (1976).
2. A. A. Dolzhkovoi, N. N. Popov, and V. P. Radchenko, "Solution of the stochastic boundary-value problem of steady-state creep for a thick-walled tube using the small parameter method," *J. Appl. Mech. Tech. Phys.*, **47**, No. 1, 134–142 (2006).
3. V. A. Kuznetsov, "Creep of stochastically nonuniform media under conditions of a plane stress state", in: *Mathematical Physics* (collected scientific papers), Kuibyshev Polytekh. Inst., Kuibyshev (1977), pp. 69–74.
4. N. N. Popov and Yu. P. Samarin, "Stress fields close to the boundary of a stochastically inhomogeneous half-plane during creep," *J. Appl. Mech. Tech. Phys.*, No. 1, 149–154 (1988).
5. N. N. Popov and Yu. P. Samarin, "Spatial problem of stationary creep of a stochastically inhomogeneous medium," *J. Appl. Mech. Tech. Phys.*, No. 2, 296–301 (1985).
6. V. P. Radchenko and Yu. A. Eremin, *Rheological Deformation and Fracture of Materials and Structural Members* [in Russian], Mashinostroenie-1, Moscow (2004).
7. A. A. Sveshnikov, *Applied Methods of the Theory of Random Functions* [in Russian], Nauka, Moscow (1968).
8. S. A. Shesterikov (ed.) *Regularity of Creep and Long-Term Strength: Handbook* [in Russian], Mashinostroenie-1, Moscow (1983).